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BOUNDARY CONDITIONS IN THE ASYMPTOTIC THEORY FOR BEAMS OF RECTANGULAR CROSS-SECTION

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1. INTRODUCTION

It is well known [1] that solutions of the improved engineering beam vibration theories suggested by Timoshenko [2], Cowper [3], Stephen and Levinson [4], Berdichevsky and Kvashnina [5], and by one of the authors [6] may give errors of lower asymptotic order than known corrections for rotary inertia and shear deformation. Although these errors are often numerically small [7], disregarding the end effects makes engineering beam theories asymptotically incorrect. The proper boundary conditions which implicitly take into account the end effect solutions (inner solutions) for a static asymptotic beam theory were obtained by Fan and Widera on the basis of the reciprocal theorem [8]. For three of the four classic types of loading and fixing conditions, this method provides explicit relations between the beam theory variables at the end without consideration of the inner solutions; however, in the case of prescribed displacement data, the authors did not show the final results, although all necessary stress distributions, in particular, for a circular cylinder were presented.

In what follows, the boundary conditions for free symmetric bending vibrations of rectangular beams with a clamped end are derived on the basis of the Hellinger–Reissner variational principle by using the eigenfunctions for the semi-infinite strip [9, 10]. The influence of the corrections for the end effects in the displacement boundary conditions on the two lowest natural frequencies is demonstrated for a uniform cantilever beam.

2. BOUNDARY CONDITIONS IN THE ASYMPTOTIC BEAM THEORY

2.1. *General equations*

Vibration equations for a linear-elastic beam of a uniform narrow rectangular cross-section of height $2H$ may be written in the Cartesian system xy as the following relations of the two-dimensional theory of elasticity for generalized plane stress [11]:

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = \rho\ddot{U}, \qquad \partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y = \rho\ddot{V},$$

$$\partial U/\partial x = (\sigma_x - v\sigma_y)/E, \qquad \partial V/\partial y = (\sigma_y - v\sigma_x)/E, \qquad \partial U/\partial y + \partial V/\partial x = \tau_{xy}/G. \quad (1)$$

Here U, V are the displacements in the x and y directions, σ_x, σ_y, τ_{xy} are the stresses, G and ρ are the shear modulus and the density, $E = 2G(1+v)$, v is Poisson's ratio, and $\dot{()} = \partial()/\partial t$. In what follows, which is confined to free symmetric bending vibrations, the classic types of boundary conditions are considered,

$$0 < x < L, \qquad y = \pm H\colon \sigma_y = 0, \qquad \tau_{xy} = 0, \quad (2)$$

$$x = e, \qquad |y| \leqslant H\colon \sigma_x = \tau_{xy} = 0, \qquad \sigma_x = V = 0, \qquad \tau_{xy} = U = 0 \qquad \text{or} \qquad U = V = 0, \quad \text{(3a–d)}$$

where $e = 0$ or $e = L$.

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The governing equations, the displacements $U = U_b$, $V = V_b$ and the stresses $\sigma_x = \sigma_b$, $\tau_{xy} = \tau_b$ of the asymptotic beam theory [6] are

$$\partial v/\partial x = -u + 6Q/5AG, \qquad \partial M/\partial x = Q + \rho I\ddot{u},$$
$$\partial Q/\partial x = \rho A\ddot{v} - \nu\rho\ddot{M}/5E, \qquad \partial u/\partial x = M/EI + \nu\rho\ddot{v}/5E, \tag{4}$$

$$U_b = yu - (2+\nu)y(y^2 - 3H^2/5)Q/6EI, \qquad V_b = v - \nu(y^2 - H^2/5)M/2EI,$$
$$\sigma_b = yM/I, \qquad \tau_b = (H^2 - y^2)Q/2I, \tag{5}$$

where u and v are the averages of U and V over the cross-section, the weighting factors y/I and $(H^2 - y^2)/2I$ being used, Q and M are the shear force and the bending moment, $A = 2H$, $I = 2H^3/3$. From equations (5), it can be seen that the boundary conditions (3a–c) may be satisfied at the end if one applies, respectively, $M = Q = 0$, $M = v = 0$, and $Q = u = 0$; however, in the case of displacement data prescribed over the end (3d), the end effect solutions of equations (1) and (2) should be considered in addition to the right-hand parts of equations (5). For long wavelength vibrations of the beam these solutions can be constructed, with the use of the eigenfunctions for the semi-infinite strip $x \geqslant 0$, $|y| \leqslant H$ [9, 10], as

$$U_n = H\{[2\varkappa_n/(1+\nu) - \gamma_n]\sin\gamma_n\eta + \varkappa_n\gamma_n\eta\cos\gamma_n\eta\}\exp(\gamma_n\xi)/2G,$$
$$V_n = -H\{[(1-\nu)\varkappa_n/(1+\nu) + \gamma_n]\cos\gamma_n\eta + \varkappa_n\gamma_n\eta\sin\gamma_n\eta\}\exp(\gamma_n\xi)/2G,$$
$$\sigma_n = \gamma_n[(2\varkappa_n - \gamma_n)\sin\gamma_n\eta + \varkappa_n\gamma_n\eta\cos\gamma_n\eta]\exp(\gamma_n\xi),$$
$$\tau_n = \gamma_n[(\varkappa_n - \gamma_n)\cos\gamma_n\eta - \varkappa_n\gamma_n\eta\sin\gamma_n\eta]\exp(\gamma_n\xi),$$

where $\sigma_n = \sigma_{xn}$, $\tau_n = \tau_{xyn}$, $\xi = x/H$, $\eta = y/H$, $\varkappa_n = \tan\gamma_n$, and $\gamma_n = -a_n \pm b_n\,\mathrm{i}$ ($\mathrm{i} = \sqrt{-1}$) are complex roots of the transcendent equation

$$\sin 2\gamma_n = 2\gamma_n, \tag{6}$$

$a_n > 0$, $b_n > 0$, $n \geqslant 1$, in particular, $a_1 = 3{\cdot}749$, $b_1 = 1{\cdot}384$.

In order to formulate boundary conditions at the ends separately, at each of them one can omit the end effect solutions arising near the opposite end which are multiplied here by the small factor $\exp(-a_n L/H)$ equalling 0·0006, even if $n = 1$ and $2H/L = 1$. Then one can write, e.g., at the end $x = 0$ (below in all functions of x and ξ, $x = \xi = 0$ was added, and $\sigma_x = \sigma$, $\tau_{xy} = \tau$ were also denoted),

$$U = \eta Hu - (2+\nu)\eta(\eta^2 - 3/5)H^3Q/6EI + \mathrm{Re}\sum_{n=1}^{\infty} A_n U_n, \tag{7a}$$

$$V = v - \nu(\eta^2 - 1/5)H^2M/2EI + \mathrm{Re}\sum_{n=1}^{\infty} A_n V_n, \tag{7b}$$

$$\sigma = \eta HM/I + \mathrm{Re}\sum_{n=1}^{\infty} A_n\sigma_n, \qquad \tau = (1-\eta^2)H^2Q/2I + \mathrm{Re}\sum_{n=1}^{\infty} A_n\tau_n, \tag{7c, d}$$

where A_n are unknown complex constants, Re () denotes the real part, and $\gamma_n = -a_n + b_n\,\mathrm{i}$ in U_n, V_n, σ_n and τ_n.

2.2. *Boundary conditions for the clamped end*

In beam theory non-engineering boundary conditions for the clamped end $x = 0$ at which $U = 0$, $V = 0$, may be obtained, in accordance with equations (7a, b), by eliminating the coefficients A_n from the following equations:

$$\eta Hu - (2 + \nu)\eta(\eta^2 - 3/5)H^3Q/6EI + \mathrm{Re}\sum_{n=1}^{\infty} A_n U_n = 0,$$

$$v - \nu(\eta^2 - 1/5)H^2M/2EI + \mathrm{Re}\sum_{n=1}^{\infty} A_n V_n = 0. \qquad (8)$$

Equations of this type have been treated by Johnson and Little [12] in the plane problem for the semi-infinite strip using the Galerkin method with bi-orthogonal weighting functions. However, as Spence [13] showed, the resulting matrix of the system of linear algebraic equations is not diagonally dominant, and for a large number of terms in the sums (8) the convergence of A_n breaks down. These computational difficulties may be surmounted in stress analysis problems by using special weighting functions as was suggested by Spence [13], but this method cannot be directly applied to beam theory.

Here the boundary conditions for the clamped end are obtained on the basis of the Hellinger–Reissner variational principle. In the variational equation (1) of reference [6] the term corresponding to the clamped end $x = 0$ is $\langle U\,\delta\sigma + V\,\delta\tau\rangle = 0$, where $\langle\ \rangle$ is an integral of η from -1 to 1. Taking a number N of terms in the sums (7) and varying the complex coefficients $A_1, \ldots, A_n$ and the values of M and Q in $\delta\sigma$ and in $\delta\tau$, one can derive a system of non-homogeneous linear algebraic equations of order $2N + 2$ for $A_1, \ldots, A_n$, u and v, assuming Q and M to be parameters. Thus, the values of the kinematic variables u and v at the end $x = 0$ and, by analogy, at the end $x = L$ are presented in unified linear forms with real coefficients λ_{11}, λ_{12}, λ_{21}, λ_{22} and upper signs for the end $x = 0$, lower signs for the end $x = L$ as

$$u = (\pm\lambda_{11}\,M + \lambda_{12}\,HQ)H/EI, \qquad v = (\lambda_{21}\,M \pm \lambda_{22}\,HQ)H^2/EI, \qquad (9)$$

the engineering boundary conditions for the clamped end are [6]

$$u = 0, \qquad v = 0. \qquad (10)$$

Table 1 shows the convergence of λ_{ij} $(i, j = 1, 2)$ to limit values as N increases, $\nu = 0{\cdot}3$. The number of coinciding figures in the coefficients λ_{12} and λ_{21} in equations (9) is not less

TABLE 1

The coefficients λ_{11}, λ_{12}, λ_{21} *and* λ_{22} *in equations* (9), *for different numbers N of terms in the sums* (7); $\nu = 0{\cdot}3$

N	$-\lambda_{11}$	$\lambda_{12} = \lambda_{21}$	$-\lambda_{22}$
1	0·00464	0·00599	0·02695
2	0·00796	0·00719	0·02820
5	0·00990	0·00773	0·02876
10	0·01048	0·00801	0·02893
20	0·01071	0·00813	0·02899
30	0·01077	0·00816	0·02901
40	0·01080	0·00818	0·02902
50	0·01081	0·00818	0·02902

than 10 for all $N \leqslant 50$ and is in accordance with the accuracy of computing, indicating the numerical stability of the above method; the complex roots γ_n of equation (6) have been determined by using the procedure suggested by Hillman and Salzer [14]. Note that as N increases, so all coefficients λ_{ij} in equation (9) vary monotonically, and their values for $N = 50$ are

$$\lambda_{11} = -0{\cdot}01081, \qquad \lambda_{12} = \lambda_{21} = 0{\cdot}00818, \qquad \lambda_{22} = -0{\cdot}02902. \tag{11}$$

The above equations of the non-engineering beam theory may be applied to the plane strain problem for an infinite uniform unloaded plate, the two opposite clamped sides of which are statically displaced one from one another by the shear force; the exact elasticity solution of this problem was obtained by Wan [15]. Substituting $v_p/(1 - v_p)$ and $E_p/(1 - v_p^2)$, where v_p and E_p are Poisson's ratio and Young's modulus for the plate, for v and E transform equations (4) into Reissner's plate theory equations and, based on the results presented in reference [15], one may determine the shear rigidity of the plate of $v_p = 1/3$, $2H/L = 0{\cdot}5$ with the errors 0·1 or 4·1% depending on which of the boundary conditions at the sides $x = 0$ and $x = L$ are used: equations (9), where E is replaced by $E_p/(1 - v_p^2)$ and λ_{ij} for $v = (1/3)/(1 - 1/3) = 0{\cdot}5$ are as $\lambda_{11} = -0{\cdot}02828$, $\lambda_{12} = \lambda_{21} = 0{\cdot}01250$, $\lambda_{22} = -0{\cdot}03232$, or the conditions (10) of Reissner's plate theory.

The boundary conditions (9) and (10) also give close results in dynamic problems: in particular, for $2H/L \leqslant 1$, $v = 0{\cdot}3$ the two lowest eigenfrequencies of a cantilever beam increase by only 1·7 and 1·2%, respectively, if one takes equations (4) with the non-engineering boundary conditions instead of the engineering ones; hereto, the frequencies of the Timoshenko beam with the shear coefficient $K = (5 + 5v)/(6 + 5v)$ [16] almost always lie between the above corresponding eigenfrequencies.

3. CONCLUSION

In the present note the governing equations of the asymptotic beam theory derived in reference [6] for uniform beams are complemented by the appropriate boundary conditions. For the three classic types of homogeneous conditions at the end (3a–c), the boundary conditions for a beam coincide with known exact equations [8]; in a case of the displacement data at the end, the above non-engineering boundary conditions also lead to results being in accordance with known solutions [15] and the qualitative estimations [17]. It should be noted, however, that the above mathematically justified asymptotic beam theory and the classic Timoshenko beam equations give practically equal eigenfrequencies.

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